The Duality and the Failure of LQG Control

The Inevitable Trade-off Between Performance and Robustness

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1. Introduction
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System Model

Consider a n-th order linear time-invariant (LTI) discrete-time dynamic system with m-dimensional input and p-dimensional output:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + \omega_k, \quad & \omega_k &\sim \mathcal{N}(0, W_k) \\ y_k &= Cx_k + v_k, & v_k &\sim \mathcal{N}(0, V_k) \end{aligned}$$

- $x_k \in \mathbb{R}^n$: state vector at time step k
- $u_k \in \mathbb{R}^m$: control input vector at time step k
- $y_k \in \mathbb{R}^p$: measurement vector at time step k
- $A \in \mathbb{R}^{n\times n}$: state transition matrix
- $B \in \mathbb{R}^{n\times m}$: control input matrix
- $C \in \mathbb{R}^{p\times n}$: observation matrix

Controllability

A LTI system is said to be **controllable** if,

$$\forall x_0, x^*, \exists k > 0, \mathbf{u}_k = [u_{k-1}, \cdots, u_1, u_0], \quad \text{such that} \quad x_k = x^*.$$

This is equivalent to $\operatorname{rank}(M_c) = n$, where $M_c = [B, AB, A^2B, \ldots, A^{n-1}B] \in \mathbb{R}^{n \times nm}$ is the controllability matrix.

$$\begin{aligned}
x_n &= Ax_{n-1} + Bu_{n-1} \\
&= A(Ax_{n-2} + Bu_{n-2}) + Bu_{n-1} \\
&= A^2 x_{n-2} + ABu_{n-2} + Bu_{n-1} \\
&= A^n x_0 + A^{n-1}Bu_0 + \cdots + ABu_{n-2} + Bu_{n-1} \\
&= A^n x_0 + M_c \mathbf{u}_n
\end{aligned}$$

$$\mathbf{u}_n = M_c^\top (M_c M_c^\top)^{-1} (x^* - A^n x_0)$$

Observability

A LTI system is said to be **observable** if,

$$\forall x_0 \in \mathbb{R}^n \exists k > 0, \mathbf{y_k} = [y_0, y_1, \dots, y_{k-1}]^\top \Rightarrow x_0.$$

This is equivalent to $\mathrm{rank}(M_o) = n$, where
$M_o = [C^\top, (CA)^\top, (CA^2)^\top, \dots, (CA^{n-1})^\top]^\top \in \mathbb{R}^{np \times n}$ is the observability matrix.

$$\begin{aligned} y_0 &= Cx_0 \\ y_1 &= Cx_1 = CAx_0 \\ &\vdots \\ y_{n-1} &= CA^{n-1}x_0 \end{aligned} \quad \Rightarrow \quad \mathbf{y}_n = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x_0 = M_o x_0$$

$$x_0 = (M_o^\top M_o)^{-1} M_o^\top \mathbf{y}_n$$

Optimal Estimator: Kalman Filter

Goal:

$$\min_{\hat{x}_{k|k}} \mathbb{E}[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^\top \mid y_1, \ldots, y_k]$$

Solution:

$$\begin{aligned}
\hat{x}_{k|k-1} &= A\hat{x}_{k-1|k-1} + Bu_{k-1} \\
\hat{P}_{k|k-1} &= A\hat{P}_{k-1|k-1}A^\top + W_{k-1} \\
K_k &= \hat{P}_{k|k-1}C^\top(C\hat{P}_{k|k-1}C^\top + V_k)^{-1} \\
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_k(y_k - C\hat{x}_{k|k-1}) \\
\hat{P}_{k|k} &= \hat{P}_{k|k-1} - K_kC\hat{P}_{k|k-1} = (\hat{P}_{k|k-1}^{-1} + C^\top V_k^{-1}C)^{-1}
\end{aligned}$$

Optimal Regulator: LQR

Goal:

$$\min_{\{u_k\}} \mathbb{E}\left[x_N^\top Q_N x_N + \sum_{k=0}^{N-1}(x_k^\top Q_k x_k + u_k^\top R_k u_k)\right]$$

Solution:

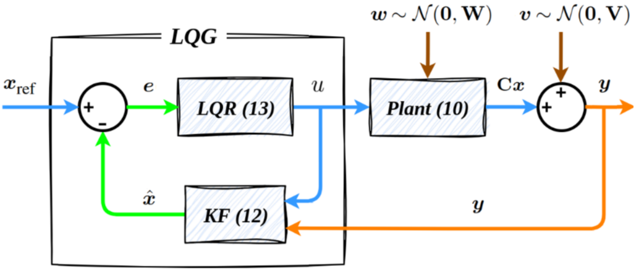
$$\begin{aligned} S_N &= Q_N \\ L_k &= (R_k + B_k^\top S_{k+1} B_k)^{-1} B_k^\top S_{k+1} A_k \\ S_k &= Q_k + A_k^\top S_{k+1}(A_k - B_k L_k) \\ u_k &= -L_k x_k \end{aligned}$$

Linear Quadratic Gaussian (LQG)

The **separation principle** states that the design of the optimal controller and the optimal observer can be separated. The optimal control law is given by:

$$u_k = -L_k \hat{x}_{k|k}$$

where $\hat{x}_{k|k}$ is the state estimate provided by the Kalman filter.



Courtesy: Daniel Engelsman

The Duality in Control Theory

Controllability vs Observability For the original system $\Sigma = (A, B, C)$, the dual system is defined as $\Sigma^* = (A^\top, C^\top, B^\top)$.

- Σ is controllable $\Leftrightarrow$ Σ^* is observable
- Σ is observable $\Leftrightarrow$ Σ^* is controllable

Controller vs Observer

- Feedback controller $u_k = -L_k x_k$ "suppresses" the state deviation x_k through inputs
- State observer $\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(y_k - C\hat{x}_{k|k-1})$ "corrects" the state estimate $\hat{x}_{k|k}$ through measurements
- The design of L_k and K_k are dual problems

The Duality in LQR and Kalman Filter

Optimization formulation of LQR:

$$\min_{x_{1:N}, u_{1:N-1}} x_N^\top Q_N x_N + \sum_{k=0}^{N-1} \left[x_k^\top Q_k x_k + u_k^\top R_k u_k \right]$$

Optimization formulation of Kalman Filter:

$$\min_{x_{1:N}, \omega_{1:N-1}} (x_0 - \hat{x}_{0|0})^\top P_0^{-1} (x_0 - \hat{x}_{0|0}) + \sum_{k=0}^{N-1} \left[(y_k - Cx_k)^\top V_k^{-1} (y_k - Cx_k) + \omega_k^\top W_k^{-1} \omega_k \right]$$

subject to $x_{k+1} = Ax_k + Bu_k + \omega_k$.

Duality:

$$A \leftrightarrow A^\top, \quad B \leftrightarrow C^\top, \quad Q \leftrightarrow W, \quad R \leftrightarrow V$$

The Duality in LQR and Kalman Filter (Cont.)

Riccati Equation in LQR:

$$\begin{cases} L_k = (R_k + B_k^\top S_{k+1} B_k)^{-1} B_k^\top S_{k+1} A_k \\ S_k = Q_k + A_k^\top S_{k+1}(A_k - B_k L_k) \end{cases}$$

$$S = A^\top SA + Q - A^\top SB(B^\top SB + R)^{-1} B^\top SA$$

Riccati Equation in Kalman Filter:

$$\begin{cases} \hat{P}_{k|k-1} = A\hat{P}_{k-1|k-1}A^\top + W_{k-1} \\ K_k = \hat{P}_{k|k-1} C^\top (C\hat{P}_{k|k-1} C^\top + V_k)^{-1} \\ \hat{P}_{k|k} = \hat{P}_{k|k-1} - K_k C\hat{P}_{k|k-1} = (\hat{P}_{k|k-1}^{-1} + C^\top V_k^{-1} C)^{-1} \end{cases}$$

$$P = APA^\top + W - APC^\top (CPC^\top + V)^{-1} CPA^\top$$

Duality:

$$A \leftrightarrow A^\top, \quad B \leftrightarrow C^\top, \quad Q \leftrightarrow W, \quad R \leftrightarrow V, \quad S \leftrightarrow P$$

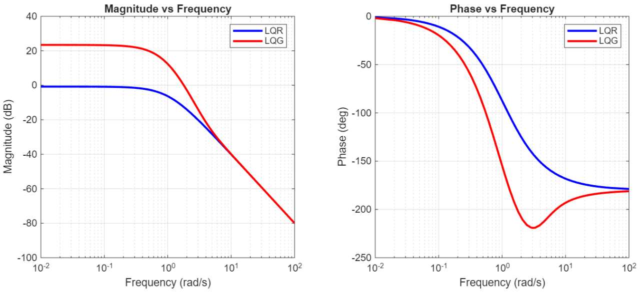
The Paradox of Optimality

LQR Robustness (SISO systems):

- ≥ 60 deg Phase Margin
- ≥ 6 dB Gain Margin
- Infinite gain reduction margin

Kalman Filter Robustness:

- Dual robustness properties at sensor output
- Excellent margins against sensor errors



The Fundamental Trade-Off

LQR's Need for High-Gain Feedback:

- Large **Q** & Small **R**
- Excellent stability margins

KF's Need for High-Gain Feedback:

- Large **W** & Small **V**
- Prompt response to new measurements

*Optimizing for individual robustness leads to a **fragile** combined LQG system.*

The Destructive Feedback Loop:

1. High-gain **L** reacts aggressively to state deviations
2. High-gain **K** amplifies sensor noise
3. This creates a **positive feedback** loop
4. Resulting in potential **instability** of the system

No stability guarantee for imperfect models, leading to the development of **H_∞ Control**

The 1970s. Lack of robustness

Figure 1: F-8C crusader aircraft

Figure 2: Trident submarine

Courtesy: NASA, Peter Suciu

Thank you for listening !

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